(a)
$$\begin{vmatrix} s_1+s_4 & s_4+s_1 & s_3+s_5 & s_5+s_3 & s_2+s_6 & s_6+s_2 \\ s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\ s_2+s_5 & s_5+s_2 & s_1+s_6 & s_6+s_1 & s_3+s_4 & s_4+s_3 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3+s_6 & s_6+s_3 & s_2+s_4 & s_4+s_2 & s_1+s_5 & s_5+s_1 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

(b), (c)
$$\begin{vmatrix} s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ 2s_4 & 2s_5 & 2s_6 & 2s_1 & 2s_3 & 2s_2 \\ 2s_5 & 2s_6 & 2s_4 & 2s_2 & 2s_1 & 2s_3 \\ 2s_6 & 2s_4 & 2s_5 & 2s_3 & 2s_2 & 2s_1 \end{vmatrix}$$

(d)
$$\begin{vmatrix} s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ s_4-s_1 & s_5-s_3 & s_6-s_2 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\ s_5-s_2 & s_6-s_1 & s_4-s_3 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\ s_6-s_3 & s_4-s_2 & s_5-s_1 & s_3-s_6 & s_2-s_4 & s_1-s_5 \end{vmatrix}$$

(e)
$$\begin{vmatrix} 2(s_1+s_4) & 2(s_3+s_5) & 2(s_2+s_6) & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ 2(s_2+s_5) & 2(s_1+s_6) & 2(s_3+s_4) & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ 2(s_3+s_6) & 2(s_2+s_4) & 2(s_1+s_5) & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ 0 & 0 & 0 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\ 0 & 0 & 0 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\ 0 & 0 & 0 & s_3-s_6 & s_2-s_4 & s_1-s_5 \end{vmatrix}$$

If H' is conjugate to H in G, the corresponding factor Θ'_1 is the same as Θ_1. For, the element in the ith row and jth column of Θ_1 is Σs_α, where s_α runs through the operators of $s_i H s_j^{-1}$. If $s_r H s_r^{-1} = H'$, the element in the ith row and jth column of Θ_1 is the same as the element in the kth row and lth column of Θ'_1, where $s_i = s_k s_r$, $s_j^{-1} = s_r^{-1} s_l^{-1}$, since $s_i H s_j^{-1} = s_k s_r H s_r^{-1} s_l^{-1} = s_k H' s_l^{-1}$. Moreover, s_j fixes s_l, and therefore the elements of the jth column of $\mathfrak{S}_1$ are the same, except as to arrangement, as those of the lth column of Θ'_1. Likewise the elements of the ith row of Θ_1 are the same, except as to arrangement, as those of the kth row of Θ'_1. Therefore $\Theta'_1 = \pm \Theta_1$.

A GENERAL METHOD OF DEDUCING THE EQUATION OF A TANGENT TO A CURVE.

By PROFESSOR G. W. GREENWOOD, M. A., McKendree College.

The following method of finding the equation of a tangent to a curve is more general than those usually given in texts in elementary analytical geometry, but can readily be substituted for any of them. At the same time it has many additional advantages.

Through any point $P\equiv(x', y')$ draw a line l making an angle θ with OX. If this line has a point, or points, in common with a locus given by a rational integral equation, denote such a point by Q. If $PQ=r$, the coördinates of Q are

$$x'+r\cos\theta, \quad y'+r\sin\theta.$$

Since Q is on the locus, its coördinates satisfy the equation of the locus. Substituting, and arranging in ascending powers of r, we have

$$0=u_0+ru_1+r^2u_2+\ldots\ldots \qquad (a)$$

If $u_0=0$, one value of r is zero; that is, one position of Q is coincident with P, or, the point P is on the locus.

Another value of r will be zero and the line l is said to have two points in common with the locus at P, if θ be chosen so that $u_1=0$. The line l is then said to be a *tangent* at P. [Notice that we do not have Q to coincide with P, and then speak of the tangent as a line through two *coincident* points; neither do we have Q approach P, and define the tangent as the limit of the line PQ. We are not troubled with the idea of limits.] The equation of l is then

$$y-y'=(x-x')\tan\theta,$$

where θ has the value which makes $u_1=0$. [If $\theta=90^\circ$, the equation of l is $x-x'=0$.]

EXAMPLE I. Find the equation of the tangent at the point $P\equiv(x', y')$ to the locus $y^2=4ax$. If a line l through P making an angle θ with OX have a point Q in common with the locus its coördinates

$$x'+r\cos\theta, \quad y'+r\sin\theta,$$

where $PQ=r$, satisfy the equation of the locus.

Consequently $0=(y'^2-4ax')+2r(y'\sin\theta-2a\cos\theta)+r^2\sin^2\theta$.

Since P is on the locus,

$$y'^2-4ax'=0. \qquad (1)$$

Hence one value of r is zero for all values of θ. Another value will be zero, that is, the line will be a tangent, if $y'\sin\theta - 2a\cos\theta = 0$. The equation of the tangent is therefore

$$(y-y')y' = (x-x')2a,$$

or, by using (1),

$$yy' = 2a(x+x').$$

If $u_0 = 0$ and $u_1 = 0$ for all values of θ, then two values of r are zero for any position of l through the point P, which is then called a *double point* on the locus. When this is the case, the line l is called a tangent when θ is chosen so that $u_2 = 0$, thus making one more value of r zero.

EXAMPLE II. Find the tangent at the origin to the locus $x^2 - y^2 - 3y^3 = 0$. If a line l through the origin making an angle θ with OX have a point Q in common with the locus, its coördinates,

$$r\cos\theta, \quad r\sin\theta,$$

where $PQ = r$, satisfy the equation.

Thus $r^2(\cos^2\theta - \sin^2\theta) - 3r^3\sin^3\theta = 0$.

Two values of r are zero for all values of θ. Hence the origin is a double point of the locus. Another value of r will be zero, that is, the line l will be a tangent, if $\cos^2\theta - \sin^2\theta = 0$; that is, if $\tan\theta = \pm 1$. We get therefore two positions of l satisfying the required condition, its equation being $y = x$ or $y = -x$.

EXAMPLE III. If an equation of the second degree represents a locus with a double point, it is the equation of two straight lines. Denote the equation by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and assume that it has a double point $P \equiv (x', y')$. Through P draw a line l making an angle θ with OX, and denote any point common to the line and the locus by Q. The coördinates of Q, viz.,

$$x' + r\cos\theta, \quad y' + r\sin\theta,$$

where $PQ = r$, satisfy the equation of the locus.

Hence $0 = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c + 2r[(ax' + hy' + g)\cos\theta + (hx' + by' + f)\sin\theta] + r^2[a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta]$.

Since at a double point two values of r must be zero, for all values of θ, we must have

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0, \tag{1}$$

$$ax' + hy' + g = 0, \tag{2}$$

$$hx' + by' + f = 0. \tag{3}$$

Multiplying (2) by x', (3) by y', and subtracting their sum from (1), we have

$$gx'+fy'+c=0. \tag{4}$$

Eliminating x', y' from (2), (3), (4), we have

$$\begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix} =0,$$

which is the condition that the given equation represents two straight lines.

EXAMPLE IV. Find the tangent at the origin to the locus $x^2+2y^2+3y^3=0$. Proceeding as in example II, we get

$$r^2(\cos^2\theta+2\sin^2\theta)+3r^3\sin^3\theta=0.$$

Two values of r are zero, so that the origin is a double point on the locus. No value of θ can be found which will make another value of r zero. Hence the locus has no real tangent at the origin. Such a point is called a *conjugate* point, and if we were to trace the curve we would find that we could not find another point satisfying the equation and as near as we pleased to the origin. [While it is true that when there are no tangents at a point on a locus, it is a conjugate point, it does not hold conversely that at a conjugate point tangents are not obtained by the usual methods. For example, we obtain $y=0$ as the tangent at the origin to the locus $y^2=2x^2y+x^4y-2x^4$, although the origin is a conjugate point.]

If $u_0=0$, and the value of θ which makes $u_1=0$ also makes $u_2=0$, but does not make $u_3=0$, the equation gives three zero values of r instead of two for the position of l given by $u_1=0$. Such a point is called a *point of inflexion*.

EXAMPLE V. Find the tangent at the origin to the locus $x^3+x+y=0$. Proceeding as in example II we get

$$0=r(\cos\theta+\sin\theta)+r^3\sin^3\theta.$$

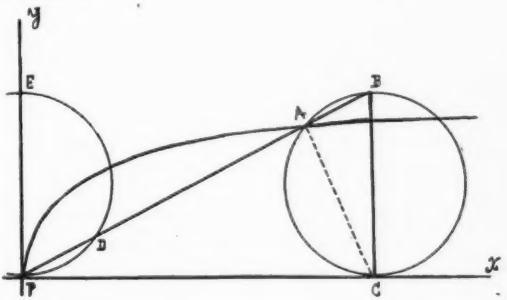
If θ be chosen so that $\cos\theta+\sin\theta=0$, we get *two* additional zero values for r. Hence the origin is a point of inflexion, the tangent at that point being $x+y=0$.

A LINKAGE FOR THE KINEMATIC DESCRIPTION OF A CISSOID.

By JOHN JAMES QUINN, Ph. D.

THEOREM. *If a line pivoted on an axis be drawn to the extremity of the diameter of a constant circle, the locus of its intersection with the circumference of the circle as the latter rolls along the axis is a cissoid.*

PROOF. Let BC be the diameter of the rolling circle, at any point in the axis PX as C; PB the line pivoted at P and intersecting the circle at the point A. Draw EP parallel to BC; then connect E and B. Hence,



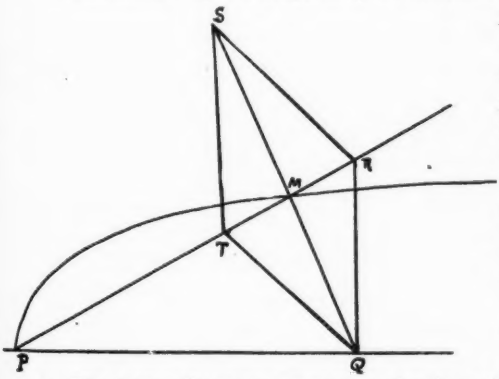
$$EB \parallel PC, \quad \angle EBA = \angle BPC,$$
$$PD = AB.$$

Hence the locus of A is a cissoid.

COROLLARY. *If from the moving vertex of a rectangle having one side constant, a perpendicular be dropped upon the varying diagonal the locus of the foot of the perpendicular is a cissoid.*

For, in rectangle $EBCP$ the side BC is constant, PB the diagonal, and AC is perpendicular to PB.

THEOREM. *If one diagonal of a rhombus be produced and pivoted at a fixed point in an axis, and if the rhombus be moved along the axis in such a way that one side is constantly perpendicular to it, the locus of the intersection of the diagonals is a cissoid.*



PROOF. Let $QRST$ be the rhombus; RQ perpendicular to PQ; the diagonal TR produced and pivoted at P.

Then since QR is constant, $\angle RMQ$ is a right angle, and the locus of M is a cissoid.

The latter theorem suggests a simple method of constructing a linkage for describing a cissoid by continuous motion.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

254. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n^2}{(16n^2-1)^2}$ beginning with $n=1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\sum_{n=1}^{\infty} \frac{n^2}{(16n^2-1)^2} = \tfrac{1}{64} \Sigma \left[\frac{1}{(4n^2-1)^2} + \frac{1}{4n-1} - \frac{1}{4n+1} + \frac{1}{(4n-1)^2} \right]$$

$$= \tfrac{1}{64} \left[\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} \cdots\cdots \right) + \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} \cdots\cdots \right) \right]$$

$$= \tfrac{1}{64} \left[\left(\frac{\pi^2}{8} - 1 \right) + \left(1 - \frac{\pi}{4} \right) \right] = \tfrac{1}{64} \left(\frac{\pi^2}{8} - \frac{\pi}{4} \right).$$

Also solved by G. W. Greenwood.

255. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Let f be the binary cubic $a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_1^2+a_3x_2^3$, $\triangle=(f,f)_2$ the covariant, the second transvectant of f over itself, and $R=2[4(a_0a_2-a_1^2) \times(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2]=(\triangle, \triangle)_2$=the second transvectant of $\triangle$ over itself. Then if $\triangle_{\kappa\lambda}$ is the $\triangle$ covariant for the cubic pencil $\kappa f+\lambda Q$, Q being the first transvectant of f over $\triangle$ we have $\triangle_{\kappa\lambda}=(\kappa^2-\frac{1}{2}\lambda^2 R)\triangle$.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

If $R\neq 0$, we may reduce the quantic to the form mX^3+nY^3.
Hence, $\triangle=2mnXY$, $R=-2m^2n^2$, $Q=m^2nX^3-mn^2Y^3$,

$$\kappa f+\lambda Q=(\kappa m+\lambda m^2 n)X^3+(\kappa n-\lambda mn^2)Y^3,$$

$$\text{and } \triangle_{\lambda\kappa}=2mn(\kappa^2-\lambda^2 m^2 n^2)XY=(\kappa^2-\tfrac{1}{2}\lambda^2 R)\triangle.$$

If $R=0$, we may reduce the quantic to the form $3lX^2Y$.
Then $\triangle=-2l^2X^2$, $Q=2l^3X^3$, $\kappa f+\lambda Q=3\kappa lX^2Y+2l^3\lambda X^3$.
Hence $\triangle_{\lambda\kappa}=-2\kappa^2l^2X^2=\kappa^2\triangle$.

Also solved by M. E. Graber.

CALCULUS.

215. Proposed by PROFESSOR B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Prove that, if the differential equation $cydx-(y+a+bx)dy-nx(xdy-ydx)=0$, be transformed into an equation between u and x by the substitution $u(y+a+bx+nx^2)=y(c+nx)$, then the variables are separable; and reduce the equation to the form $dv/\phi(v)=dx/\phi(x)$ by the further substitution $v=au+\beta$, a and β being suitably determined. *Euler.* [Forsyth's *Differential Equations*, p. 48, Ex. 4.]

Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

The first equation may be written

$$\frac{dy}{dx}=\frac{(c+nx)y}{y+a+bx+nx^2}. \quad \text{Thus } \frac{dy}{dx}=u,$$

and as $u(y+a+bx+nx^2)=(c+nx)y$, we have by differentiating with respect to x, writing u for $\frac{dy}{dx}$, and $\frac{u(a+bx+nx^2)}{c+nx-u}$ for y, and re-arranging,

$$\frac{du}{[c^2-bc+na+u(b-2c)+u^2]u}=\frac{dx}{(a+bx+nx^2)(c+nx)}.$$

This is of the form $\frac{du}{f(u)}=\frac{dx}{\phi(x)}$.

Let $u=c+nv$, then $du=ndv$, $f(u)=n(a+bv+nv^2)(c+nv)$. Hence,

$$\frac{dv}{(a+bv+nv^2)(c+nv)}=\frac{dx}{(a+bx+nx^2)(c+nx)}$$

which is of the form $\frac{dv}{\phi(v)}=\frac{dx}{\phi(x)}$.

Also solved by W. W. Landis, and G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

132. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Disregarding the order of λ, μ, ν, how many sets of solutions has the congruence $\lambda+\mu+\nu\equiv 0 \pmod{p-1}$ (p prime)? [A. E. Western.]

*Solution by the PROPOSER.

Let n_i be the number of solutions in which i of the numbers λ, μ, ν are equal. If $p\equiv 1 \pmod 3$ $n_3=3$, the solutions being

*See problems for solution, *Diophantine Analysis*, No. 134.

$$\lambda \equiv \mu \equiv \nu \equiv 0, \text{ or } \frac{p-1}{3} \text{ or } \frac{2(p-1)}{3} \pmod{p-1}.$$

If $p \equiv 2 \pmod 3$, $n_3 = 1$, viz., $\lambda \equiv \mu \equiv \nu \equiv 0 \pmod{p-1}$. Next when two are equal so that the congruence is $\lambda + 2\mu \equiv 0 \pmod{p-1}$, μ cannot be $\equiv 0, \frac{p-1}{3}$, or $\frac{2(p-1)}{3}$ for then it would be $\equiv \lambda$. With these exceptions μ can have any value, and for each value of μ the congruence gives one value of λ. Hence if $p \equiv 1 \pmod 3$, $n_2 = p-4$, and when $p \equiv 2 \pmod 3$, $n_2 = p-2$.

If account be taken of the order of λ, μ, ν the total number of solutions of the congruence is $(p-1)^2$ since each unknown may assume $p-1$ values and when two are fixed the other is determined by the congruence. In terms of n_1, n_2 and n_3 the above totality of solutions equals $6n_1 + 3n_2 + n_3$. Hence $6n_1 + 3n_2 + n_3 = (p-1)^2$, so that if $p \equiv 1 \pmod 3$, $n_1 = \frac{1}{6}(p^2 - 5p + 10)$, and when $p \equiv 2 \pmod 3$, $n_1 = \frac{1}{6}(p^2 - 5p + 6)$. Finally, when $p \equiv 1 \pmod 3$, the total number (N) of solutions, disregarding order, of the given congruence is

$$N = 3 + p - 4 + \tfrac{1}{6}(p^2 - 5p + 10) = \tfrac{1}{6}(p^2 + p + 4),$$

and if $p \equiv 2 \pmod 3$,

$$N = 1 + p - 2 + \tfrac{1}{6}(p^2 - 5p + 6) = \tfrac{1}{6}(p^2 + p).$$

GEOMETRY.

278. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

AF, MN are parallel lines indefinitely extended toward FN; at right angles to AF, MN is AM of length 22; upon the base AB, which is in line with AM, is the triangle ABC whose sides are $AB = 21$, $BC = 10$, $AC = 17$; find the sides of the maximum similar triangle with base extending from B to some point in AF, the vertex in line with MN.

Solution by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

Let D be the vertex in AF, E the vertex in MN, $AD = x$, $ME = y$. Two cases are possible, BD the short side or BD the long side. The latter gives the maximum. Let $BD = 21M$, $DE = 10M$, $BE = 17M$. Then

$$441M^2 = 441 + x^2 \quad \ldots\ldots (1),$$
$$289M^2 = 1 + y^2 \quad \ldots\ldots (2),$$
$$100M^2 = (x-y)^2 + 484 \quad \ldots\ldots (3).$$

Substitution of x, y from (1) and (2) in (3) gives $\sqrt{[(M^2-1)(289M^2-1)]} = 15M^2 + 1$. Hence $M^2 = 5$, $M = \sqrt{5}$, and the sides are $21\sqrt{5}$, $17\sqrt{5}$, $10\sqrt{5}$.

Also solved by the Proposer.

280. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

On any diameter of a given ellipse is taken a point such that the tangents from it intercept on the tangent at one end of the diameter a length equal to the diameter; the ellipse being $a^2y^2+b^2x^2-a^2b^2=0$. Prove that the locus of the point is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$.

Solution by the PROPOSER.

Let the diameter through any point $P(x_1, y_1)$ of the locus cut the ellipse in A, A'. The chord RS of contact of P and the tangent at A' are parallel to the diameter at A_1A_2 conjugate to AA': and then

$$\frac{PB}{RS}=\frac{PA'}{TQ} \quad \ldots\ldots (1),$$

B being the intersection of RS and PA'. The equation to RS is

$$y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1} \quad \ldots\ldots (2).$$

If $y=mx+c$ ..... (3), is the equation to any chord of $a^2y^2+b^2x^2=a^2b^2$ ..... (4), the length of chord is $l=2ab\sqrt{[(1+m^2)(m^2a^2+b^2-c^2)]}\div(m^2a^2+b^2)$ ..... (5). Comparing (2) and (3), $m=-\frac{b^2x_1}{a^2y_1}$ ..... (6), $c=\frac{b^2}{y_1}$ ..... (7). Substituting (6) and (7) in (5), we have

$$RS=2\sqrt{[(a^4y_1^2+b^4x_1^2)(b^2x_1^2+a^2y_1^2-a^2b^2)]}\div(a^2y_1^2+b^2x_1^2) \quad \ldots\ldots (8).$$

OP or $y=\frac{y_1}{x_1}x$ ..... (9) cuts the curve in

$$x'=-\frac{abx_1}{\sqrt{(a^2y_1^2+b^2x_1^2)}},\quad y'=-\frac{aby_1}{\sqrt{(a^2y_1^2+b^2x_1^2)}} \quad \ldots\ldots (10).$$

(9) cuts (2) in B, or

$$x''=\frac{abx_1}{\sqrt{(a^2y_1^2+b^2x_1^2)}},\quad y''=\frac{aby_1}{\sqrt{(a^2y_1^2+b^2x_1^2)}} \quad \ldots\ldots (11).$$

Then $\overline{PB}^2=(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2-a^2b^2)^2\div(a^2y_1^2+b^2x_1^2)^2$ ..... (12),
$\overline{PA'}^2=(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2+a^2b^2)^2\div(a^2y_1^2+b^2x_1^2)$ ..... (13),
$\overline{TQ}^2=4a^2b^2(x_1^2+y_1^2)\div(a^2y_1^2+b^2x_1^2)$ ..... (14).

Squaring (1), substituting (8), (12), (13), and reducing, we have

$$a^2b^2(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2-a^2b^2)$$
$$=(a^4y_1^2+b^4x_1^2)(\sqrt{[a^2y_1^2+b^2x_1^2]}+ab)^2 \quad \ldots\ldots (15),$$

the locus required. Equation (15) may be factored by the solution of Algebra problem 250, the relevant factor, with subscripts omitted, being

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Also solved by G. B. M. Zerr.

281. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

In the proposition in solid geometry "If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane of the lines," it is assumed that two intersecting lines have a common perpendicular. Prove it.

Solution by DR. GEORGE BRUCE HALSTED, Gambier, Ohio.

To prove that two intersecting straights have a common perpendicular.

To each of the given straights, a, b, at their intersection point A construct a perpendicular plane (Halsted's *Rational Geometry*, §337). They have a straight in common (H. R. G. I 6), which is perpendicular to a and perpendicular to b by H. R. G. §333, and that without *assuming* that any two intersecting straights have a common perpendicular.

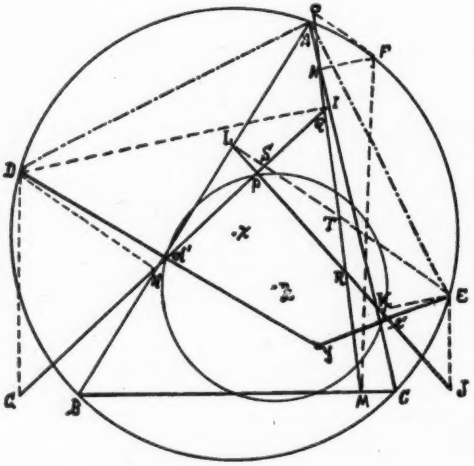
Also solved by A. H. Holmes, Rev. J. H. Meyer, and G. B. M. Zerr.

282. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The pedal lines of any two points on the circum-circle of a triangle concur in an angle equal to that subtended by the said points.

Solution by the PROPOSER.

Upon the circum-circle of triangle ABC take, first, any diameter DE. Then its pedal lines GHI and JKL will concur at P in a right angle.



Join DA, EA, and let GHI cut the perpendicular EL to AB in S. Then EKA and ELA being right angles, $EKLA$ are concyclic about EA; and angle $EAK=ELK$. Similarly, $DHIA$ are concyclic about DA; and $ADI=AHI$. Therefore the right angled triangles AID and SLH are similar; with angles DAI and LSP equal. And hence, even as the angles DAI and EAK are complementary, being within the semi-circle DAE; so also are their equals SLP and LSP. So that LPS is a right angle.

Next, take any other point F upon the circumcircle; and let its pedal line MNO meet those of D and E in Q and R, respectively. Join AF; and let MNO cut the perpendicular EL to AB in T, even as GHI cut it in S.

Then angle DAI has been shown equal to LSP; and hence also to QST. While EL and FO being parallel, QTS is equal to QOF. But $FNAO$ being concyclic about ON, QOF and FAN are equal; while angle DAF is compounded of DAI and FAN. So that GQO, or $QST+QTS$ are equal to DAF; and thus their supplement GQM to the angle subtended by arc DAF.

And lastly; since PRQ is the complement to PQR, even as the angle subtended by arc FE is to that subtended by arc DF; the said angles PRQ and that subtended by arc FE must be also equal. And similarly for any point upon the circum-circle.

Corollary. DE being a diameter of the circum-circle, the medial points,—say u, v, and w,—of the sides AB, BC, and CA of the inscribed triangle must also bisect the segments LH, GJ, and KI cut off upon said sides by its pedal lines GHI and JKL. Each bisecting perpendicular,—say xu, xv, and xw,—from x, the circum-center, being parallel to DH and EL, DG and EJ, and DI and EK, respectively. And thus the said medial points u, v, and w are always the circum-centers of the right angled triangles HPL, GPJ, and KPI, respectively.

Also solved by G. W. Greenwood, and J. Scheffer.

283. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The right angled intersection of the pedal lines of any diameter of the circumcircle lies on the "nine points circle" of the inscribed triangle.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let x [see figure above] be the circum-center, y the ortho-center, and z the center of the nine points circle of triangle ABC; passing, by construction, through the feet of the perpendiculars from the ortho-center upon the sides of the triangle, and also through the medial points of the lines joining the ortho-center to the vertices of the triangle.

Then, by known theorems, the center of z of the nine points circle bisects xy, the distance between the ortho- and circum-centers. While the said lines from the orth-ocenter perpendicular to the sides of the triangle cut its circum-circle at twice their distance to these sides. So that the ortho-center Y is evidently also a center of similitude to the nine points and circum-circles; with all magnitudes of the latter half those of the former.

And hence d', where DY, by a known theorem, is bisected by the pedal line GHI of D, is also its intersection by the nine points circle. And similarly, e', the middle point of YE, lies both on the pedal line JKL of E, and on the nine points circle.

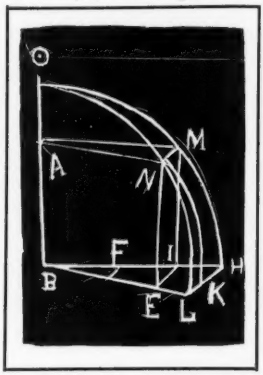
Therefore $d'e'$ is parallel to DE, and bisects XY in Z, and is thus a diameter of the nine points circle; even as DXE is of the circum-circle. So that $d'Pe'$ being a right angle, P lies on the nine points circle.

273. Proposed by A. H. HOLMES, Brunswick, Maine.

Required a purely geometrical solution of the problem, to find the contents of a solid generated by the revolution of a semi-segment of a circle about the sine of its arc.

Solution by the PROPOSER.

HBO is a quadrant whose revolution about *BO* as an axis generates a hemisphere. *BAF* is a semi-segment of radius=*BO*. Draw *AM* parallel to *HB* and *MI* parallel to *BO*. Suppose the quadrant to revolve about its axis a very small distance, the point *H* moving to *L* so as to generate *HBOLB*, *M* falling on *N*. Through *NE* pass a plane parallel to *HBO*. The semi-segment *HMI*=*AFB* generates *HIELMN*; of which the part *EKLN*=part generated by *BAF*.



It is obvious that the volume generated by the semi-segment *BFA* in an entire revolution will equal that generated by *HMI minus* the sum of the solids *HIKEMN* lying about the circumference of the base of the hemisphere.

But *MNHIEK*=*IE*×area of the semi-segment and the entire sum of all these is equal to the circumference described by *BI* as radius into the same area. If we put *BO*=*r*, *BE*=*c*, *BA*=*s*, and arc *AF*=*a*, we obtain for the solid generated by *BOMI*, $\frac{2\pi}{3}(s\,c^2+r^3-r^2 s)$. Consequently, for the solid generated by *HIM*, $\frac{2\pi}{3}(r^2 s-s\,c^2)$. The sum of all the solids *HKEINM* = semi-segment *MHI*×$2\pi c$ $=\pi(c\,a\,r-s\,c^2)$. Consequently the volume sought is

$$=\frac{2\pi}{3}(r^2 s-s\,c^2)-\pi(c\,a\,r-s\,c^2)=\pi\left(\frac{sc^2}{3}+\frac{2r^2 s}{3}-c\,a\,r\right).$$

Putting $c^2=r^2-s^2$, this becomes $\pi(s\,r^2-s^3/3-c\,a\,r)$.

GROUP THEORY.

14. Proposed by O. E. GLENN, Springfield, Mo.

Hölder has proved* that any group (G) of order $\sum_{i=1}^{n} p_i$ (p_i a prime$\neq p_j$) may be generated as follows: $M^\mu = N^\nu = 1$, $N^{-1}MN=M^\alpha$, where $\{M\}$ is the product of all the invariant subgroups of G of prime order and $\{N\}$ is any one of a set of conjugate cyclical subgroups of order ν, ($\sum_{i=1}^{n} p_i=\mu\nu$). Find the generating relations of G in terms of operations of prime order, and express M and N in terms of these operations, for $n=4$.

*See Burnside, *Theory of Groups*, p. 353.

Solution by the PROPOSER.

The group (G) contains a subgroup H_{pqr} of order pqr self-conjugately. This subgroup may be any one of the six known types

(1) $\{P\}\{Q\}\{R\}$, (2) $\{P\}\{Q, R\}$, (3) $\{Q\}\{P, R\}$, (4) $\{R\}\{Q, P\}$,
(5) $\{P, QR\}=[P^p=Q^q=R^r=1,\ QR=RQ,\ R^{-1}PR=P^\beta\ ,\ Q^{-1}PQ=P^\alpha\ ]$,
(6) $\{P, Q, R\}=[P^p=Q^q=R^r=1,\ QP=QP,\ R^{-1}PR=P^\beta\ ,\ R^{-1}QR=Q^\gamma\ ]$,

where by $\{\Omega\}$ is understood the cyclic group of order ω and $\{A, B\}=[A^a=B^b=1,\ B^{-1}AB=A^\theta\ ]$. $\theta^b\equiv 1 \pmod a$.

Let N_t be the number of subgroups of order t in H. Then if

I) $N_p=1$, $N_q=1$, $N_r=1$, H is the type (1), and if S be any operation of G of order s we have $S^{-1}PS=P^\epsilon$, $S^{-1}QS=Q^\kappa$, $S^{-1}RS=R^\lambda$ where ϵ, κ, λ, may be part or all unity or primitive roots of the congruences $x^s\equiv 1 \pmod{p, q, r}$. Classification according to these values gives the following distinct types of G additional to the one abelian type [1, 1, 1, 1].

$$P^p=Q^q=R^r=S^s=1,\quad \varepsilon^s\equiv 1 \pmod p,\quad \kappa^s\equiv 1 \pmod q,\quad \lambda^s\equiv 1 \pmod r.$$

[1] $PQ=QP,\ PR=RP,\ QR=RQ,\ S^{-1}PS=P^\epsilon\ ,\ SQ=QS,\ SR=RS.$
[2] $PQ=QP,\ PR=RP,\ QR=RQ,\ PS=SP,\ S^{-1}QS=S^\kappa\ ,\ SR=RS.$
[3] $PQ=QP,\ PR=RP,\ QR=RQ,\ PS=SP,\ SQ=QS,\ S^{-1}RS=R^\lambda\ .$
[4] $PQ=QP,\ PR=RP,\ QR=RQ,\ S^{-1}PS=P^\epsilon\ ,\ S^{-1}QS=Q^\kappa\ ,\ SR=RS.$
[5] $PQ=QP,\ PR=RP,\ QR=RQ,\ S^{-1}PS=P^\epsilon\ ,\ SP=QS,\ S^{-1}RS=R^\lambda\ .$
[6] $PQ=QP,\ PR=RP,\ QR=RQ,\ PS=SP,\ S^{-1}QS=Q^\kappa\ ,\ S^{-1}RS=R^\lambda\ .$
[7] $PQ=PQ,\ PR=RP,\ QR=RQ,\ S^{-1}PS=P^\epsilon\ ,\ S^{-1}QS=Q^\kappa\ ,\ S^{-1}RS=R^\lambda\ .$

II) Suppose next that $N_p=1$, $N_q=1$, $N_r=q$, so that H is of type (2). By a proper choice of R we get $S^{-1}RS=R^\lambda$, and in virtue of the relations

$$(RS)^{-1}Q(RS)=Q^{\gamma\kappa}=(SR^\lambda\ )^{-1}Q(SR^\lambda\ )=Q^{\kappa\gamma^\lambda}$$

we have $\gamma\kappa(\gamma^{\lambda-1}-1)\equiv 0 \pmod q$, $\lambda\equiv 1 \pmod r$. The parameters ε and κ may assume the same series of values as in the previous case. The new types are

[8] $PQ=QP,\ PR=RP,\ R^{-1}QR=Q^\gamma\ ,\ PS=SP,\ SQ=QS,\ SR=RS.$
[9] $PQ=QP,\ PR=RP,\ R^{-1}QR=Q^\gamma\ ,\ S^{-1}PS=P^\epsilon\ ,\ SQ=QS,\ SR=RS.$
[10] $PQ=QP,\ PR=RP,\ R^{-1}QR,Q^\gamma\ ,\ PS=SP,\ S^{-1}QS=Q^\kappa\ ,\ SR=RS.$
[11] $PQ=QP,\ PR=RP,\ R^{-1}QR=Q^\gamma\ ,\ S^{-1}PS=P^\epsilon\ ,\ S^{-1}QS=Q^\kappa\ ,\ SR=RS.$

III) Let $N_p=1$, $N_q=1$, $N_r=p$. Then H is the type (3), and since at least one subgroup of order r is permutable with S, a proper choice of R gives $S^{-1}RS=R^\lambda$. Reasoning similar to above gives

$$\varepsilon\delta(\delta^{\lambda-1}-1)\equiv 0 \pmod p,\ \lambda\equiv 1 \pmod r.$$

We thus arrive at the types

[12] $PQ=PQ,\quad R^{-1}PR=P^{\delta},\quad RQ=QR,\quad PS=SP,\quad SQ=QS,\quad SR=RS.$
[13] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad RQ=QR,\quad S^{-1}PS=P^{\epsilon},\quad SQ=QS,\quad SR=RS.$
[14] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad RQ=QR,\quad PS=SP,\quad S^{-1}QS=Q^{\kappa},\quad SR=RS.$
[15] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad RQ=QR,\quad S^{-1}PS=P^{\epsilon},\quad S^{-1}QS=Q^{\kappa},\quad SR=RS.$

IV) Next suppose $N_p=1$, $N_r=1$, $N_q=p$, so that H is the type (4). Since $\{R\}$ is invariant in H it is invariant in G, so that $S^{-1}RS=R^{\lambda}$. Also a proper choice of Q,gives $S^{-1}QS=S^{\kappa}$. Then evidently

$$\alpha z(\alpha^{\kappa-1}-1)\equiv 0 \pmod{p},\quad \kappa\equiv 1 \pmod{q}$$

which gives the additional types

[16] $Q^{-1}PQ=P^{\alpha},\quad PR=RP,\quad PQ=QP,\quad PS=SP,\quad SQ=QS,\quad SR=RS.$
[17] $Q^{-1}PQ=P^{\alpha},\quad PR=RP,\quad PQ=QP,\quad S^{-1}PS=P^{\epsilon},\quad SQ=QS,\quad SR=RS.$
[18] $Q^{-1}PQ=P^{\alpha},\quad PR=RP,\quad PQ=QP,\quad PS=SP,\quad SQ=QS,\quad S^{-1}RS=R^{\lambda}.$
[19] $Q^{-1}PQ=P^{\alpha},\quad PR=RP,\quad PQ=QP,\quad S^{-1}PS=P^{\epsilon},\quad SQ=QS,\quad S^{-1}RS=R^{\lambda}.$

V) If $N_p=1$, $N_q=1$, $N_r=pq$; H of type (6), there is again a subgroup of order r of (G) permutable with an arbitrary S, $S^{-1}RS=\mathrm{R}^{\lambda}$. Hence

$$\epsilon\delta(\delta^{\lambda-1}-1)\equiv 0 \pmod{p},\quad \gamma\kappa(\gamma^{\lambda-1}-1)\equiv 0 \pmod{q},\quad \lambda\equiv 1 \pmod{r}.$$

Again there are four new types, viz.,

[20] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad R^{-1}QR=Q^{\gamma},\quad SP=PS,\quad SQ=QS,\quad SR=RS.$
[21] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad R^{-1}QR=Q^{\gamma},\quad S^{-1}PS=P^{\epsilon},\quad SQ=QS,\quad SR=RS.$
[22] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad R^{-1}QR=Q^{\gamma},\quad SP=PS,\quad S^{-1}QS=Q^{\kappa},\quad SR=RS.$
[23] $PQ=QP,\quad R^{-1}PR=P^{\delta},\quad R^{-1}QR=Q^{\gamma},\quad S^{-1}PS=P^{\epsilon},\quad S^{-1}QS=Q^{\kappa},\quad SR=RS.$

VI) Lastly, if $N_p=1$, $N_q=p$, $N_r=p$, [H of type (5)] we may by properly choosing R assume $S^{-1}RS=R^{\lambda}$. The generator of order q is not so at our choice but we may write

$$S^{-1}QS=P^{-v}Q^{\kappa}P^{v},\quad S^{-1}PS=P^{\epsilon}.$$

It follows as formerly that $\lambda\equiv 1 \pmod{r}$. Also

$$(RS)^{-1}Q(RS)=P^{-v}Q^{\kappa}P^{v}=(SR)^{-1}Q(\mathrm{SR})=P^{-v\delta}Q^{\kappa}P^{v\delta}.$$

Hence $P^{-v(\alpha^{\kappa}-1)}=P^{-v\delta(\alpha^{\kappa}-1)}$, $v(\alpha^{\kappa}-1)(\delta-1\equiv 0(\mathrm{mod}\,p)$, and since $\kappa\neq 0$, and $\alpha\neq 1$, $\delta\neq 1$, we have $v\equiv 0 \pmod{p}$ and

$(QS)^{-1}P(QS)=P^{\alpha\epsilon}=(SQ^{\kappa})P(SQ^{\kappa})=P^{\epsilon v^{\kappa}}$, so that $\kappa\equiv 1 \pmod{q}$.

Combination of the remaining relations in all possible ways reveals no inconsistencies. We thus have the fresh types

[24] $Q^{-1}PQ=P^{\alpha}$, $R^{-1}PR=P^{\beta}$, $QR=RQ$, $SP=PS$, $SQ=QS$, $SR=RS$.
[25] $Q^{-1}PQ=P^{\alpha}$, $R^{-1}PR=P^{\beta}$, $QR=RQ$, $S^{-1}PS=P^{\kappa}$, $SQ=QS$, $SR=RS$.

We next let T represent the number of distinct types contained in a given set of defining relations and tabulate T, M, N, and the relations connecting p, q, r, s for each of the 26 distinct types.

Type	T	*Arithmetical relations*	M	N
[1, 1, 1, 1]	1		$PQRS$	I
1	1	$p\equiv 1(s)$*	PQR	S
2	1	$q\equiv 1(s)$	PQR	S
3	1	$r\equiv 1(s)$	PQR	S
4	$s-1$	$p\equiv q\equiv 1(s)$	PQR	S
5	$s-1$	$p\equiv r\equiv 1(s)$	PQR	S
6	$s-1$	$q\equiv r\equiv 1(s)$	PQR	S
7	$(s-1)^2$	$p\equiv q\equiv r\equiv 1(s)$	PQR	S
8	1	$q\equiv 1(r)$	PQS	R
9	1	$p\equiv 1(s)$, $q\equiv 1(r)$	PQ	RS
10	1	$q\equiv 1(rs)$	PQ	RS
11	$s-1$	$p\equiv 1(s)$, $q\equiv 1(rs)$	PQ	RS
12	1	$p\equiv 1(r)$	PQS	R
13	1	$p\equiv 1(rs)$	PQ	RS
14	1	$p\equiv 1(r)$, $q\equiv 1(s)$	PQ	RS
15	$s-1$	$p\equiv 1(rs)$, $q\equiv 1(s)$	PQ	RS
16	1	$p\equiv 1(q)$	PRS	Q
17	1	$p\equiv 1(qs)$	PR	QS
18	1	$p\equiv 1(q)$, $r\equiv 1(s)$	PR	QS
19	$s-1$	$p\equiv r\equiv 1(s)$, $p\equiv 1(q)$	PR	QS
20	$s-1$	$p\equiv q\equiv 1(r)$	PQS	R
21	$r-1$	$p\equiv q\equiv 1(r)$, $p\equiv 1(s)$	PQ	RS
22	$r-1$	$p\equiv q\equiv 1(r)$, $q\equiv 1(s)$	PQ	RS
23	$(r-1)(s-1)$	$p\equiv q\equiv 1(rs)$	PQ	RS
24	1	$p\equiv 1(qr)$	PS	QR
25	1	$p\equiv 1(qrs)$	P	QRS

The method of determining T for a given type may be illustrated by type (23). Let $\Omega=S^xR^yQ^zP^w$ be the general operator of G. Then $\Omega^{\omega}=S^{\omega x}R^{\omega y}Q^{z}\left(\frac{\kappa^{\omega x}\gamma^{\omega y}-1}{\kappa^{x}\gamma^{\omega}-1}\right)P^{w}\left(\frac{\varepsilon^{\omega x}\delta^{\omega y}-1}{\varepsilon^{x}\delta^{y}-1}\right)$, and it follows that Ω is of order s only when y, z, and w are zero, of order r only when x, z, and w are zero, p only when x, y, z are zero, and q only when x, y, w are zero. Hence the most general transformation of G is given by $S_0=S^x$, $R_0=R^y$, $Q_0=Q$, $P_0=P$. These reproduce the type with δ, γ, ε, κ replaced by δ^x, γ^x, ϵ^y, κ^y. We may therefore fix δ and ϵ and replace γ and κ by γ^k, κ^h, respectively ($k=1, 2, \ldots, \overline{r-1}$, $h=1, 2, \ldots, \overline{s-1}$). Hence there are $(r-1)(s-1)$ types of G.

The lowest order of the form $p\,q\,r\,s$ is 210. There are 11 groups of this order.

*An abbreviation of $p\equiv 1 \pmod{s}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

259. Proposed by ARTEMUS MARTIN, M. A., Ph. D., LL. D., Washington, D. C.

On page 167 of George Bruce Halsted's *Metrical Geometry* (Mensuration), Boston, 1881, Table of Scalene Triangles, is found the following triangle, viz., Sides 21, 61, 65; Area 420. The sides of a rational scalene triangle, whose sides have no common divisor, can not all be odd; one must be even and the other two odd. It is required to find the error in the sides of the above triangle, assuming that the area is correct.

260. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The necessary and sufficient condition that a binary form be a perfect nth power is that its Hessian vanish.

261. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series, $\frac{1}{n^p}+\frac{3}{n^{2p}}+\frac{5}{n^{3p}}+\frac{7}{n^{4p}}+\frac{9}{n^{5p}}+\ldots\ldots$

AVERAGE AND PROBABILITY.

176. Proposed by T. N. HAUN, Mohawk, Tenn.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? (Problem 72, p. 503, Williamson's *Integral Calculus*.)

CALCULUS.

217. Proposed by PROFESSOR F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\underset{n=\infty}{\text{Lim}}\ \frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots(2n)}$.

218. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin mx\sin nx}{\sin x}dx$; (b) $\int_0^{\frac{1}{2}\pi}\frac{\cos mx\sin nx}{\sin x}dx$, where n is a positive integer. Also, modify the result for the case of m an integer.

219. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

In the article "Infinitesimal Calculus" in the *Encyclopaedia Britannica* Vol. XIII, page 24, I notice the following: "Of all triangular pyramids standing on a given triangular base, and of given altitude, find that whose surface is the least." A solution is required.

DIOPHANTINE ANALYSIS.

134. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

How many sets of solutions has the congruence $\lambda+\mu+\nu+\xi\equiv 0 \pmod{p-1}$ p being a prime number; the order of λ, μ, ν, ξ being disregarded.

GEOMETRY.

284. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose that two radii R and r, whose center is the origin, revolve with uniform angular velocities 3θ and θ, respectively. What is the equation of the locus of P, the projection parallel to the X axis of the extremity of the radius r on the radius R produced if necessary.

b) Apply this curve to the trisection of an angle.

c) Suppose the ratio of their velocities is $n\theta:\theta$. Show how we can effect the multisection of an angle.

285. Proposed by G. E. BROCKWAY, Nashua, N. H.

Prove without the aid of the circle, that if the bisectors of the angles of a triangle be drawn, the greatest bisector falls on the least side.

286. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

On the sides of a given triangle measure off equal distances from the extremities of the base, and at these points erect perpendiculars to the sides. Find the locus of the point of intersection of these perpendiculars. Solve by methods of analytic geometry.

287. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the points whose abscissae are 0, $a\sqrt{3}$, and $-a\sqrt{3}$ are points of inflexion on the locus $x^2y-a^2x+a^2y=0$.

MISCELLANEOUS.

157. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in the Educational Times, London.)

An inelastic rod 9 feet long is placed with its upper end upon a rough vertical plane and its lower end on a smooth horizontal plane, and so that it makes an angle of 45° with each plane. It is now released, and strikes against a smooth sphere of 1 foot diameter placed in contact with the two planes. Determine the subsequent motion.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Group Theory, 8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the

eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

Miscellaneous, 151. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Sum the series $\sum_{r=1}^{r=m} \operatorname{cosec}\left[\frac{2r-1}{4m}\pi + \theta\right]\operatorname{cosec}\left[\frac{2r-1}{4m}\pi - \theta\right]$.

NOTES AND NEWS.

Mr. George Brett has been elected tutor in mathematics in the College of the City of New York.

The roll of foreign members of the Circolo Mathematico Di Palermo includes the names of forty-five Americans.

The list of active members just issued by the London Mathematical Society contains the names of thirty American mathematicians.

Father J. G. Hagen, S. J., professor of astronomy in Georgetown University, has been offered the directorship of the Vatican Observatory.

At its December meeting the Chicago Section of the American Mathematical Society elected officers for the year 1906, as follows: Chairman, Professor Alexander Ziwet; Secretary, Professor H. E. Slaught; additional member of the Executive Committee, Professor A. G. Hall.

Probably the oldest journal devoted to *elementary* mathematics is the *Maandelykse Mathematische Liefhebbery*, 1754—1769. The John Crerar Library in Chicago, has the complete set of seventeen volumes of this journal, which was published at Purmerende in the Netherlands.

Professor James Mills Pierce, Perkins professor of mathematics and astronomy at Harvard University, has presented his resignation to take effect a year hence. Professor Pierce's service at Harvard covers a period of fifty-two years, he having been appointed tutor in the University in 1854.

Dr. J. J. Quinn of Warren, Pa., is preparing a monograph on the trisection of the angle. It is to contain a large variety of solutions of the historic problem by higher plane curves and by analytic methods, with historical notes. He desires that all who have suggestions as to methods of dealing with the problem should send them to him. For these suggestions proper credit will be given.

The current number of the *Proceedings of the London Mathematical Society* contains the following contributions: "On the arithmetical nature of the coefficients in a group of linear substitutions of finite order" (second paper), by Pro-

fessor W. Burnside; "The continuum and the second number class," by Mr. G. H. Hardy; "On 'well-ordered' aggregates," by Professor A. C. Dixon; "On the arithmetic continuum," by Dr. E. W. Hobson; "On some difficulties in the theory of transfinite numbers and order types," by Mr. B. Russell; "On the Hessian configuration and its connection with the group of 360 plane collineations," by Professor W. Burnside; "On the representation of certain asymptotic series as convergent continued fractions," by Professor L. J. Rogers.

Dr. Hendrik Antoon Lorentz, professor of mathematical physics in the University of Leiden, has begun a course of lectures at Columbia University on the theory of electrons and it applications to the phenomena of light and radiant heat. The program is as follows:

Friday, March 23, 4 to 6 p. m.; Saturday, March 24, 10 to 12 a. m.; and Friday, March 30, 4 to 6 p. m.—General principles; theory of free electrons.

Saturday, March 31, 10 to 12 a. m., and Friday, April 6, 4 to 6 p. m.—Emission and absorption of heat.

Saturday, April 7, 10 to 12 a. m.; Wednesday, April 11, 4 to 6 p. m.; and Thursday, April 12, 4 to 6 p. m.—The Zeeman effect. Propagation of light in ponderable bodies.

Thursday, April 26, 4 to 6 p. m., and Friday, April 27, 4 to 6 p. m.—Optical phenomena in moving systems.

The following were elected members of the American Mathematical Society at its February meeting in New York: Mr. M. J. Babb, University of Pennsylvania; Mr. William Betz, East High School, Rochester, N. Y.; Mr. G. D. Birkhoff, University of Chicago; Mr. W. D. Breuke, Harvard University; Mr. B. E. Carter, Massachusetts Institute of Technology; Dr. H. L. Coar, University of Illinois; Miss Anna Johnson, Harvard University; Mr. W. D. Lambert, U. S. Coast Survey; Mr. W. A. Luby, Central High School, Kansas City, Mo.; President W. J. Milne, New York State Normal College; Professor Richard Morris, Rutgers College; Mr. W. J. Newlin, Harvard University; Miss R. A. Pesta, Wendell Phillips High School, Chicago, Ill.; Dr. H. B. Phillips, University of Cincinnati; Mr. A. R. Schweitzer, University of Chicago; Mr. C. G. Simpson, Michigan College of Mines; Mr. A. W. Stamper, Columbia University; Mr. F. C. Touton, Central High School, Kansas City, Mo.; Mr. M. O. Tripp, College of the City of New York.

The following periodicals have been received: The Scientific American, The Educational Times, The Nation, The Review of Reviews, The Literary Digest, Ohio Educational Monthly, The Ohio Teacher, The Physical Review, Bulletin of American Mathematical Society, School Science and Mathematics, Proceedings of the London Mathematical Society, The Open Court, The School Visitor, Popular Astronomy, L'Enseignement Mathematique, Bollettino della Associazione, Bob Taylor's Magazine, The University Herald.

FOR THE METRIC SYSTEM

The following is the text of a bill which has recently been introduced by Representative Littauer to fix the standard of weights and measures in the United States by the adoption of the metric system:

Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled:

That from and after the first of July, nineteen hundred and eight, all of the Departments of the Government of the United States, in the transaction of business requiring the use of weight and measurement, shall employ and use the weights and measures of the metric system.

The committee of publicity of the American Metrological Society, of which Professor Simon Newcomb is chairman, has sent out a circular letter in support of the bill. Some quotations follow:

"Notwithstanding some recent misleading statements to the contrary, made by opponents of the bill, the Metric System during the past thirty years has made the most substantial and important progress of its history. By the establishment of the International Bureau of Weights and Measures in 1872, the Metric System became in the fullest sense an International System. Its subsequent introduction into actual and general use in Germany and the neighboring countries have given it the character of a real International System, and secured for it a commanding position which neither the British nor any other system ever possessed, and which make it as near a permanent institution as any human arrangement can be. At the same time it is among English speaking people themselves, the medium in which all scientific research is carried on, the system in which all electrical measurements are made, and in which all higher education is given, for which reason thousands of our young people are already acquainted with it.

Under present conditions the British system is an ugly excresence on the world's literature and practical arts which the general welfare requires we should abolish as speedily as possible. Already the conflict of two systems is a seriou obstacle to international trade and a hindrance to international coöperation in every direction. The sentiment in favor of the Metric System is so far advanced in the British Empire that it is a question whether we will not be anticipated in its adoption. The expressions of Boards of Trade, educational bodies, and Colonial Governments leave no doubt but that England would immediately follow us in the adoption of the Metric System should we be fortunate enough to first take the step.

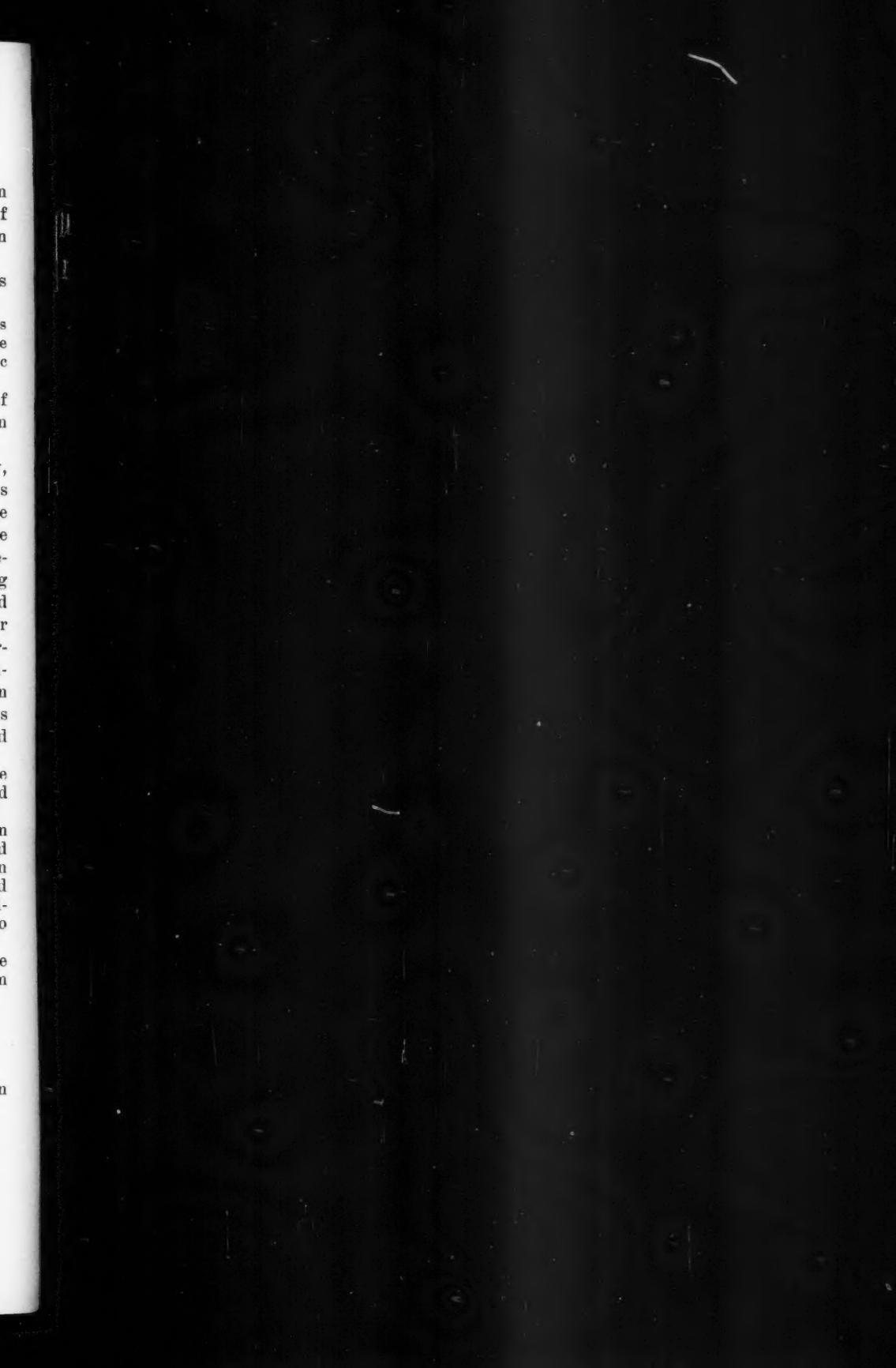
For these reasons, among others, we earnestly request you to obtain the largest possible expression of opinion favorable to the introduction of the system into all Government work by Act of Congress."

ERRATA.

Page 27, line 11 from bottom, for 15 read 13.

Page 18, lines 5 and 6 from bottom. The result of substituting in Lagrange's formula should read,

$$v = \frac{1}{x}\left[1 + \frac{\log_e x}{x} + \frac{3}{2!}\left(\frac{\log_e x}{x}\right)^2 + \ldots + \frac{n^{n-2}}{(n-1)!}\left(\frac{\log_e x}{x}\right)^{n-1} + \ldots\right]$$



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A GENERALIZED TRIGONOMETRIC SOLUTION OF THE CUBIC EQUATION.

By W. D. LAMBERT, Coast and Geodetic Survey.

The trigonometric solution of the cubic, as commonly given, is either limited to the "irreducible case," or treated by a special method for each case. The following solution treats all cases by the same device, and if tables of hyperbolic functions are at hand, is no less convenient for numerical calculation than other methods. Nor is it restricted to real coefficients, a fact which may sometimes make it convenient in applications of the theory of functions.

As a preliminary, consider the sine, $p+iq$, belonging to a complex angle $u+iv$, *i. e.*, let

$$\sin(u+iv)=p+iq=re^{ai}. \qquad (1)$$

By expanding, and separating real and imaginary parts, we find that

$$\sin u\,\mathrm{Cosh}v=p,\quad \cos u\,\mathrm{Sinh}v=q. \qquad (2)$$

Hence,

$$\frac{p}{\sin u}=\mathrm{Cosh}v,\quad \frac{q}{\cos u}=\mathrm{Sinh}v. \qquad (3)$$

Squaring each equation in (2), and subtracting, we find that

$$\frac{p^2}{\sin^2 u}-\frac{q^2}{\cos^2 u}=\mathrm{Cosh}^2 v-\mathrm{Sinh}^2 v=1. \qquad (4)$$

By substituting $\cos^2 u=1-\sin^2 u$ in (4), and solving the resulting equation as a quadratic in $\sin^2 u$, we get